# The Zero Divisor Graph of the Ring ($Z_{2^2 p}$)

Nazar H. Shuker[1] and Payman A. Rashed[2]

[1]Department of Mathematic, College of Computer Sciences and Mathematic, University of Mosul, Mosul, Iraq

[2]Department of Mathematic, College of Basic Education, University of Salahaddin, Erbil, Kurdistan Region, Iraq

***Abstract*—In this paper, we consider the crossing number and the chromatic number of the zero divisor graph $\Gamma(Z_{2^2 p})$ to show that this type of zero divisor graphs is bipartite graph, and the smallest cycle in $\Gamma(Z_{2^2 p})$ is of length four this implies that the girth is equal four.**

***Index Terms*—Bipartite graph, crossing number, girth, planar graph, zero divisor graph of the ring ($Z_{2^2 p}$).**

## I. Introduction

The idea of zero divisor graph was first introduced by I. Beck in (1988), followed by Anderson and Livingston (1999), and some others Coykendall, et al., (2012) and Shuker, et al., (2012), showed that the zero divisor graph by Γ(*R*) for commutative ring R, is always simple connected graph.

A planar graph is a graph that can be drawn on the plane in such a way that its edges are not intersecting each other (Harary, 1969). The crossing number *Cr*(*G*) of a graph *G* is the minimum number of edge crossings of a plane drawing of the graph *G* according to Sankar and Sankeetha (2012). In graph theory, a planar graph is a graph that can be embedded in the plane, i.e. it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other's; such a drawing is called a plane graph or planar embedding of the graph. A plane graph can be defined as a planar graph with a mapping from every node to a point on a plane, and from every edge to a plane curve on that plane, such that the extreme points of each curve are the points mapped from its end nodes, and all curves are disjoint except on their extreme points (Harary, 1969).

In 2012, Sankar and Sankeetha posted two open conjectures, (*i*) For any graph $\Gamma(Z_{pq})$, where $p$ and $q$ are distinct prime numbers with $p < q$, then $Cr\Gamma(Z_{pq}) = (p-1)(p-3)(q-1)(q-3)/16$, (ii) for any graph $\Gamma(Z_{p^2})$, where $p$ is any prime then $Cr\Gamma\left(Z_{p^2}\right) = (p-1)(p-3)^2(p-5)/64$. Furthermore in 2013, Malathi, et al., compare the rectilinear crossing number with the crossing number thereby forming an inequality by different conjectures.

In this environment, we study and address crossing number and chromatic number of the zero divisor graph $Z_{2^2 p}$ in which *R* is a commutative ring and *Z*(*R*) is the set of zero-divisors of *R*. We associate a graph Γ(*R*) to *R* with the set of vertices $Z^*(R) = Z(R) - \{0\}$, of non-zero zero divisor elements of *R* and for distinct $v, u \in Z^*(R)$, the vertices *v* and *u* are adjacent if and only if $vu = 0$. Throughout this work, we consider the ring $Z_{2^2 p}$. This paper was organized as follows: Finding the crossing number of the zero divisor graph $\Gamma(Z_{2^2 p})$ in the second section, in the third section we studied the girth, while in the four section we changed the style of zero divisor graph and proved it is bipartite graph and finally in the last section we find the chromatic number of the graph $\Gamma(Z_{2^2 p})$. While in our conclusion, we find the precise number of the crossing number of the ring $Z_{2^2 p}$. We also conclude that the chromatic number of $Z_{2^2 p}$ is equal two for all *p*.

## II. The Crossing Number of a Zero Divisor Graph $\Gamma(Z_{2^2 p})$

In this section, we consider the crossing number of the zero divisor graph of the ring $Z_{2^2 p}$.

### *A. Definition (Malathi, et al., 2013)*

The crossing number *Cr*(*G*) of a graph *G* is the lowest number (minimum number) of edge crossings of a plane drawing of the graph *G*. For instance, a graph is planar if and only if its crossing number is zero. A good drawing of a graph *G* is "good" if and only if all edges intersect at most one.

We start this section with the following example.

Example 1: If $p = 7$, then the zero divisor graph $\Gamma\left(Z_{2^2 p}\right) = \Gamma\left(Z_{2^2 7}\right) = \Gamma(Z_{28})$, and $Z^*(Z_{28}) = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 7, 21\}$, with center $C = 2P = 14$, has only two odd zero divisor elements, $p = 7$ and $3p = 21$.

The crossing number $Cr(Z_{28}) = 1 + 2 + 1 + 2 = 6$ as shown in Fig. 1.

ARO-The Scientific Journal of Koya University
Volume IV, No 2(2016), Article ID: ARO.10058, 04 pages
DOI: 10.14500/aro.10058
Received 23 October 2014; Accepted 20 September 2015
Regular research paper: Published 23 December 2016
Corresponding author's e-mail: payman.rashed@su.edu.krd

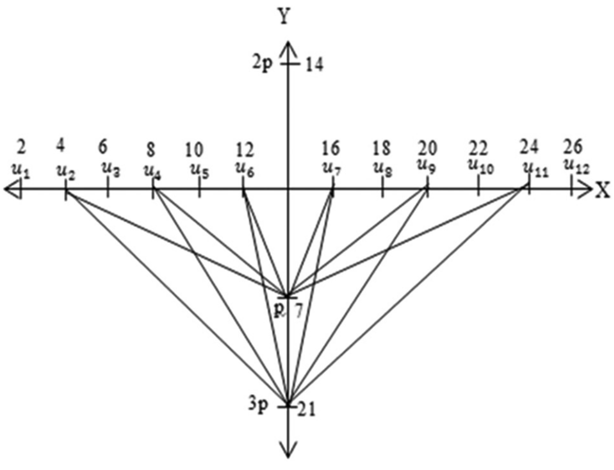


Fig. 1. Crossing number of the zero divisor graph $\Gamma(Z_{2^2 7})$

*B. Definition (Bondy and Murty, 2013)*

The degree of a vertex $v$ in the graph $G$ is the number of edges incident with $v$ in the graph and denoted by $deg(v)$.

Recall that the center of the zero divisor graph $C = 2p$.

Next, we shall give the following result.

*C. Theorem*

The crossing number of $Z_{2^2 p}$ , is $\left[\frac{p-1}{2}\frac{p-2}{2}\right]$.

Proof: The number of zero divisor elements (vertices) of the zero divisor graph $\Gamma\left(Z_{2^2 p}\right)$ is $(2p + 1)$, while the zero divisors are $\{2, 4, 6,\ldots,2p, 2p + 2,\ldots,4(p-1), p, 3p\}$, and the center of the zero divisor graph is $C = 2p$, for any prime $p$, and $deg(C) = 2p-2$.

Now, $p$ and $3p$ are two vertices in the zero divisor graph of degree $(p-1)$ which are the greatest degree of the graph except the center.

Let $u = p$ and $v = 3p$, by degree of $u$ and $v$ there exist another vertices $w_i$ distinct from $u$, $v$ such that $w_i$ is adjacent with both $u$ and $v$, then $uw_i = 0$, $vw_i = 0$ for $I = 1, 2, 3,\ldots\ldots, p-1$.

But $uv = p.3p = 3p^2$ ($3p^2$ not divide $2^2p$), this implies u and are non adjacent vertices but each $u$ and $v$ are adjacent with $p-1$ other vertices of $\Gamma\left(Z_{2^2 p}\right)$ which they are even vertices then we divide the zero divisor set $Z^*$ of $\Gamma\left(Z_{2^2 p}\right)$ into two parts $V_1 = \{u, v, w\} = \{p, 2p, 3p\}$, and $V_2 = \{2, 4, 6,\ldots,(4p-2)\}$. Clearly any vertices $u$ and $v$ in $V_1$ are nonadjacent, same as in $V_2$, where $|V_1| = 2$ and $|V_1| = 2p-1$ , since $|V| = 2p+1$, and $V(\Gamma\left(Z_{2^2 p}\right) = V_1 + V_2$, $V_1 \cap V_2 = \emptyset$.

Let $u = p$ and $v = 2$ in $V_2$, then $2^2$ does not divide $uv = 2p$, if $u = p$ and $v = 8$ then $2^2p$ divides $uv = 2^3$, implies that the vertices $u$ and $v$ are adjacent with some vertices of $V_1$ and nonadjacent with the other vertices of $V_2$.

Let $D$ be a good drawing of $\Gamma\left(Z_{2^2 p}\right)$, means it has a minimum number of crossing, the proof by method of induction on $p$.

If $p = 3$, then the crossing number is zero, if $p = 5$, then $|V| = 2p + 1 = 11$, $|V_1| = 3$ and $|V_1| = 3 = 11-3 = 8$.

Let the elements in $V_1$ be $v_1$, $v_2$, $v_3$ and elements in $V_2$ are $u_1, u_2, u_3, \ldots, u_{2p-2}$ denote by $Cr_D(v_i, u_j)$ the number of crossing of edges one terminate at $v_i$, the other at $u_j$ and by $Cr_D(v_i)$ the number of crossing edges which are terminate at $v_i$.

Clearly the crossing number $\Gamma\left(Z_{2^2 p}\right)$.

$$Cr_D(\Gamma\left(Z_{2^2 p}\right)) = \sum_{i=1}^{i=3}\sum_{j=1}^{8} Cr_D\left(v_i, u_j\right)$$

We consider the vertices of $V_1$, the proof based on having all of the vertices on X and Y axis. The first thing is to place $|V_2|/2 = \frac{p-1}{2}$ vertices on one side of X axis and $|V_1| = 3$ vertices on the Y axis, then we connect all vertices on the X axis with the vertices on Y axis. The proof based on having all of the vertices of $\Gamma\left(Z_{2^2 p}\right)$ on X and Y axis. First, we place the vertices of $V_2$ on X axis such that 4 vertices on one side of X coordinate and other vertices of $V_1$ on the Y axis in a way that $2p = C$ in the positive and $p$, $3p$ on the negative side, respectively. Then, we connect the vertices of $V_2$ with $2p$ we get no crossing and connect the vertex $p$ with the vertices of $u_i$ of $V_2$ where $u_i = 0$ mod 4 on $X$ axis also we get no crossing. Finally, when we connect the vertex $3pV$ with the vertices $u_i = 0$ mod 4, we get the crossing as follows:

$Cr(u_1) = Cr(u_3) = Cr(u_6) = Cr(u_8) = 0$

since they not adjacent with $v_1$, $v_3$

$Cr(u_2) = Cr(u_7) = 0$.

$Cr(u_4) = 1$ and $Cr(u_5) = 1$ in each side of X axis.

$Cr(\Gamma(Z_{2^2 p}) = \sum_{i=1}^{8} Cr\left(vi\right) = 2$ (1) since we have two side of X axis, then we must multiple by 2.

$$Cr(\Gamma(Z_{2^2 p}) = \left[\frac{4}{2}\right]\left[\frac{3}{2}\right] = \left[\frac{5-1}{2}\right]\left[\frac{5-2}{2}\right]$$

So $Cr(\Gamma(Z_{2^2 p}) = \left[\frac{p-1}{2}\right]\left[\frac{p-2}{2}\right]$

In general, we place $\left[\frac{2^2 p-2}{2}-1\right]$ (except the center $2p$) vertices on both side of X and Y axis such that $p$, $3p$ on one side of Y axis and $2p$ on the other side of it, see Fig. 2. The

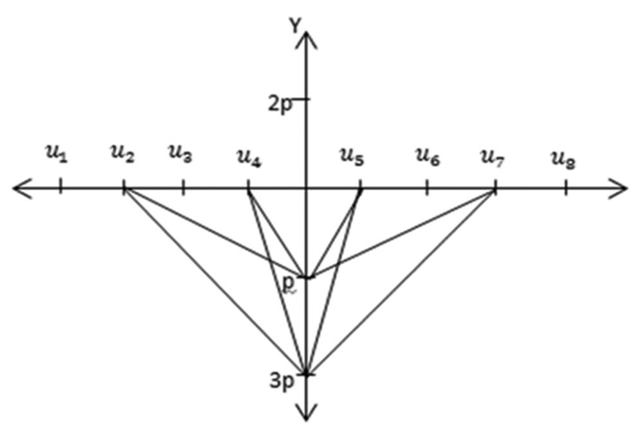


Fig. 2. Crossing number of the zero divisor graph $\Gamma(Z_{2^2 p})$

crossing happens when we connect $v_3$ with vertices $u_i$ of $V_2$ if $i$ is even, $i = 2, 4, 6, \ldots 2^2p-2$, i.e. $u_i = 0 \bmod 4$. And crossing is increase with increasing the vertices $u_i$ in each side of X axis and its depend on $p$. then the number of vertices $u_i$ adjacent with $v_3 = 3p$ are $\left\lceil \dfrac{\frac{2^2 p-2}{2}-1}{2} \right\rceil$.

$$= \left\lceil \frac{2^2 p-2}{4} - \frac{1}{2} \right\rceil = \left\lceil \frac{2^2 p-2-2}{4} \right\rceil$$

$$= \left\lceil \frac{2^2 p-4}{4} \right\rceil = \left\lceil \frac{4(p-1)}{4} \right\rceil = p-1 \text{ in each side of X axis}$$

as follow.

$Cr(u_2) = 0$, $Cr(u_4) = 1$, $Cr(u_6) = 2$ and so on in this manner in one side of X axis till.

$$Cr(u_{p-i}) = \left\lceil \frac{p-2}{2} \right\rceil$$

Therefore $Cr_D(\Gamma(Z_{2^2p})) = \sum_{i=1}^{4p-4} Cr(v_3, u_i)) = 2[1 + 2 + 3 + \ldots + \ldots]$

$$= \left\lceil \frac{p-1}{2} \right\rceil \left\lceil \frac{p-2}{2} \right\rceil$$

## III. The Girth in the Zero Divisor Graph $\Gamma(Z_{2^2p})$

In this section, we consider the girth of the zero divisor graph of the ring $\Gamma(Z_{2^2p})$.

*D. Definition (Beck, 1988)*

The girth in the graph $G$ is the smallest cycle contained in the graph. Next, we shall give the following result.

*E. Theorem*

The girth of the zero devisor graph $\Gamma(Z_{2^2p})$ is 4.

Proof: The vertices of the zero divisor graph is of the form $\{u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n, p, 2p, 3p\}$ where $2p$ is the center, $p$, $3p$ are odd zero divisors and $u_i$, $v_j$ represented the even zero divisor of the form:

$u_{i+1} = ui + 4$, $i = 1, 2, \ldots, n-1$ and $j = 0 \bmod 4$, $j = 1, 2, \ldots, n$, $vn = 4p-4$ respectively.

Since each vertex $u_i$ is of degree one, is adjacent to center only ($u_i.2p = 0$) and its not adjacent to any other vertex in $\Gamma(Z_{2^2p})$, $u_i.v_j \neq 0$, $u_i.p \neq 0$ and $u_i.3p \neq 0$, for all $i, j = 1, 2, \ldots n$, then the $deg(u_i) = 1$.

But each $v_i$ is adjacent to the odd vertices $p$, $3p$ and the center $2p$, $v_i.p = v_i.3p = v_i.2p = 0$, since $v_i = 0 \bmod 4$, then the smallest cycle in the zero divisor graph $\Gamma(Z_{2^2p})$ is as follow: Start from the vertex $2p$ to $v_m \to p \to v_k \to 2p$ for all $m, k \leq n$ then the girth of $\Gamma(Z_{2^2p}) = 4$, as shown in Fig. 3.

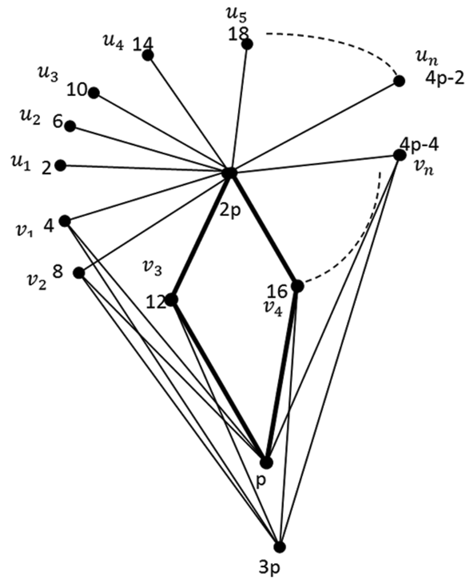


Fig. 3. The girth of $\Gamma(Z_{2^2p})$

## IV. The Partition of the Zero Divisor Graph $\Gamma(Z_{2^2p})$

*F. Definition*

The graph $G$ is said to be $n$-partite if $V$(G) divided in to subsets $V_1, V_2, .., V_n$ of vertices of the graph $G$, such that:

1. $V_i \neq \emptyset$, for $i = 1, 2., n$
2. $V_i \cap Vj = \emptyset$, for $i, j = 1, 2, \ldots, n$
3. $\cup_{i=1}^{n} V_i = V(G)$.

Provided that the vertices in each partition set $V_i$ are nonadjacent, but they can be adjacent with each other. If $n = 2$, then the graph is called bi-partite graph.

*G. Theorem*

The zero divisor graph $\Gamma(Z_{2^2p})$ is bipartite graph.

Proof: Since the vertices of zero divisor graph $\Gamma(Z_{2^2p})$ with respect to the degree of vertices is of the form $\{u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n, p, 2p, 3p\}$ and up to zero divisor relation (two vertices $a$, $b$ are adjacent if and only $ab = 0$) we divided the zero devisor set in to two subset of vertices $V_1 = \{u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n\}$ and $V_2 = \{p, 2p, 3p\}$ such that $u_i$ is adjacent with center $2p$ only, while $v_i$ is adjacent with $p$ and $3p$ respectively. However, the vertices in the set $V_1$ are nonadjacent together so the vertices of the set $V_2$ are nonadjacent with each other as shown in the Fig. 4, implies that $\Gamma\left(Z_{2^2p}\right)$ is bipartite graph.

## V. Chromatic Number of the Zero Divisor Graph $\Gamma\left(Z_{2^2p}\right)$

A coloring of a graph $G$ is a mapping $Co: V(G) \to S$. The elements of $S$ are called colors; the vertices of one color form a color class. If $|S| = k$, we say that $Co$ is a $k$-coloring (often we use $S = \{1.,k\}$). A coloring is proper if adjacent vertices have different colors.

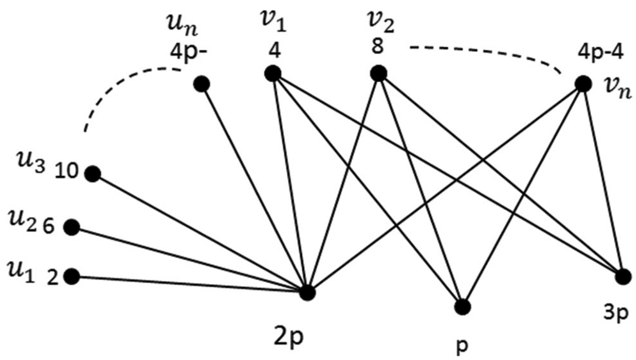


Fig. 4. The bipartite graph $\Gamma(Z_{2^2 p})$

Consider a graph $G = (V, E)$ then edge coloring: No two edges that share an endpoint get the same color. Vertex coloring: No two vertices that are adjacent get the same color use the minimum amount of colors. This is the chromatic number denoted by $\chi(G)$.

A graph is $k$-colorable if it has a proper $k$-coloring. The chromatic number $\chi(G)$ is the least $k$ such that $G$ is $k$-colorable. Obviously, $\chi(G)$ exists as assigning distinct colors to vertices yields a proper $|V(G)|$-coloring. An optimal coloring of $G$ is a $\chi(G)$-coloring, (Duana, 2006).

A graph $G$ is $k$-chromatic if $\chi(G) = k$. In a proper coloring, each color class is a stable set. Hence, a $k$-coloring may also be seen as a partition of the vertex set of $G$ into $k$ disjoint stable sets $S_i = \{v|Co(v) = i\}$ for $1 \leq i \leq k$. Therefore, $k$-colorable are also called $k$-partite graphs. Moreover, 2-colorable graphs are very often called bipartite.

Clearly, if $H$ is a subgraph of $G$ then any proper coloring of $G$ is a proper coloring of $H$. i.e., If $H$ is a sub graph of $G$, then $\chi(H) \leq \chi(G)$.

Most upper bounds on the chromatic number come from algorithms that produce colorings. The most widespread one is the greedy algorithm. A greedy coloring relative to a vertex ordering ($v_1 < < v_n$) of $V(G)$ is obtained by coloring the vertices in the order $v_1 \ldots v_n$, assigning to $v_i$ the smallest-indexed color not already used on its lower-indexed neighborhood. In a vertex-ordering, each vertex has at most $\Delta(G)$ earlier neighbors, so the greedy coloring cannot be forced to use more than $\Delta(G) + 1$ colors.

### *H. Proposition (Beck, 1988)*

$\chi(G) \leq \Delta(G) + 1$.

Next, we shall give the following result.

### *I. Theorem*

The chromatic number of $\Gamma(Z_{2^2 p})$ is 2.

i.e., $\chi\left(\Gamma\left(Z_{2^2 p}\right)\right) = 2$

Proof: Since the graph $\Gamma\left(Z_{2^2 p}\right)$ is bipartite graph (Theorem 4.2), then we use exactly two color, first color for vertices in each partite set $V_1$ and second to the other partite set $V_2$ where $V(G) = V_2$ U $V_2$, since no two vertices in $V_1$ neither in $V_2$ are adjacent together, then certainly they have the same color. Further, the total number of color used in the zero divisor graph is exactly two color, then $\chi\left(\Gamma\left(Z_{2^2 p}\right)\right)$ is two and the graph $\chi\left(\Gamma\left(Z_{2^2 p}\right)\right)$ is 2-colorable.

## VI. Conclusion

In general, the crossing number of any zero divisor graph of the ring $Z_n$, is given by inequality (less than or greater than). In this work, we found the precise number of the crossing number of the ring $Z_{2^2 p}$. We also conclude that the chromatic number of $Z_{2^2 p}$ is equal two for all $p$, and this graph can be drown as a bipartite graph up to the adjacency relation in the graph.

## References


Anderson, D.D. and Livingston, P.S., 1999. The zero divisor graph of a commutative ring. *Journal of Algebra,* 217, pp.434-447.

Beck, I., 1988. Coloring of commutative ring. *Journal of Algebra,* 116, pp208-226.

Bondy, J.A. and. Murty, U.S.R., 2013. *Graph Theory and Application North.* Holland, New York. Amsterdam. Oxford.

Coykendall, J., Wagstaff, S.S., Sheppardon, L. and Spiroff, S., 2012. On zero divisor graph. *Journal of Commutative Algebra,* 2, pp.241-299.

Duane, A., 2006. Proper coloring and p-partite structures of the zero divisor graph. *Rose Holman Undergraduate Math Journal,* 7(2), pp.1-7.

Harary, F., 1969. *Graph Theory.* Addison-Wesley Publishing Company, California.

Harju, T., 2005. *Graph theory*, Department of Mathematics, University of Turku, Finland.

Malathi, M., Sankeetha, S. and Sankar, J.R., 2013. Rectilinear crossing number of a zero divisor graph. *International Mathematical Forum*, 8(12), pp583-589.

Sankar, J.R. and Sankeetha, S., 2012. Crossing number of a zero divisor graph. *International Journal of Algebra,* 6(32), pp.1499-1505.

Shuker, N.H., Mohammad, H.Q. and Ali, A.M., 2012. The zero divisor graph of $Z_{p^n q}$. *International Journal of Algebra,* 6, pp.1049-1055.